

Complexity of some Path Problems in DAGs and Linear Orders.

Abstract. We investigate here the computational complexity of three natural problems in directed acyclic graphs. We prove their NP Completeness and consider their restrictions to linear orders.

Mathematics Subjects Classification : .

1. Introduction.

In this paper, we precise and extend the results of [1] about the complexity of path problems in directed graphs. In particular we focus ourselves on the case of DAGs and very particular ones : linear orders.

2. Problems.

Problem 1 : Null Weighted Path.

input : a DAG $G = (V, E)$ endowed with a weight function $w : E \mapsto \mathbb{Z}$ and two vertices s, t .

question : is there a directed path π from s to t such that

$$\sum_{e \in \pi} w(e) = 0$$

Proposition (1.) *The Null Weighted Path Decision Problem is NP-Complete, even on linear orders.*

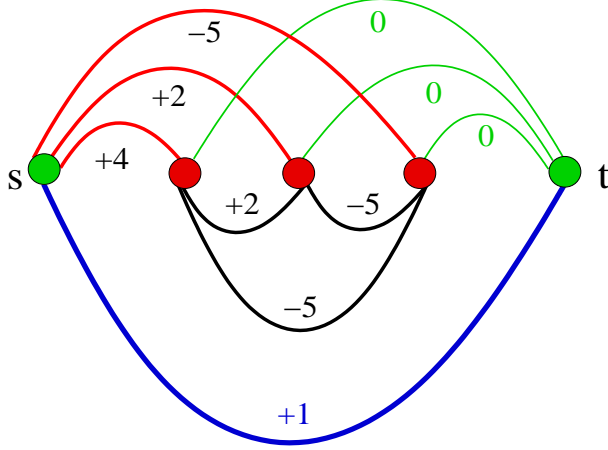
Proof. Obviously, it is in NP. To prove that it is NP hard, we reduce the SUBSET SUM Problem to it. Let $A = \{a_1, \dots, a_n\}$ be a subset of \mathbb{Z}^n . One can construct in polynomial time a DAG $G = (V, E)$, a weight function and two vertices s, t that admits a path from s to t of null weight if and only if there exists a non empty set $S \subseteq A$ such that

$$\sum_{a \in S} a = 0$$

The DAG $G = (V, E)$ has $n+2$ ordered vertices $V = [s = 0, 1, 2, \dots, n, n+1 = t]$ and the arcs are all the pairs $e_{ij} = (i, j)$ with $i < j$. Hence, G is a linear order. The weights are :

$$w(e_{ij}) = \begin{cases} a_j & \text{for } 0 \leq i < j \leq n \\ 0 & \text{for } 1 \leq i \leq n \text{ and } j = t \\ +1 & \text{for } i = 0 \text{ and } j = t \end{cases}$$

The following example shows this construction for $A = \{4, 2, -5\}$ (arcs are oriented from left to right) :



Assume there exists a non empty subset of A $S = \{a_{i_1}, \dots, a_{i_k}\}$ such that $a_{i_1} + \dots + a_{i_k} = 0$. Then, one can assume that $i_1 < i_2 < \dots < i_k$. Hence, $\pi = [s, a_{i_1}, \dots, a_{i_k}, t]$ is a null weighted path from s to t .

For the converse, observe that every path from s to t (excepted the direct arc (s, t) of weight $+1$) contains at least one arc with an element of A . Hence, a null weighted path from s to t describes a non empty subset of A with null sum. ■

Now we consider a similar problem on DAGs with positive weights.

Problem 2 : K Weighted Path.

input : a DAG $G = (V, E)$ endowed with a weight function $w : E \mapsto \mathbb{N}$, two vertices s, t and an integer $K \geq 0$

question : is there a directed path π from s to t such that

$$\sum_{e \in \pi} w(e) = K$$

Proposition (2.) *The K Weighted Path Decision Problem is NP-Complete, even on linear orders.*

Proof. Obviously, it is in NP. To prove that it is NP hard, we reduce again the SUBSET SUM Problem to it. Let $A = \{a_1, \dots, a_n\}$ be a subset of \mathbb{Z}^n . One can construct in polynomial time a DAG $G = (V, E)$, a weight function,

an integer K and two vertices s, t that admits a path from s to t of weight K if and only if there exists a non empty set $S \subseteq A$ such that

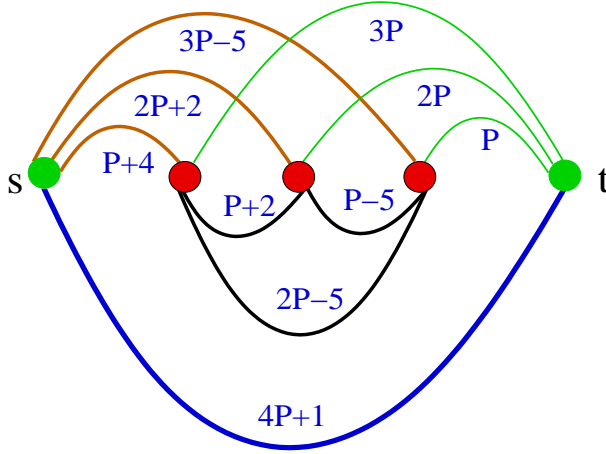
$$\sum_{a \in S} a = 0$$

Let $P = -\min(\{0\} \cup A)$ and fix $K = (n+1)P$.

The construction is similar as previously. However, one translates the weight of each arc (i, j) by $(j-i)P$. The DAG $G = (V, E)$ has $n+2$ ordered vertices $V = [s = 0, 1, 2, \dots, n, n+1 = t]$ and the arcs are all the pairs $e_{ij} = (i, j)$ with $i < j$. Hence G is a linear order. The weights are :

$$w(e_{ij}) = \begin{cases} (j-i)P + a_j & \text{for } 0 \leq i < j \leq n \\ (j-i)P & \text{for } 1 \leq i \leq n \text{ and } j = t \\ (j-i)P + 1 & \text{for } i = 0 \text{ and } j = t \end{cases}$$

For $A = \{4, 2, -5\}$ one obtains with $P = -\min(\{0, 4, 2, -5\}) = +5$ the following linear order DAG :



Assume there exists a non empty subset of A $S = \{a_{i_1}, \dots, a_{i_k}\}$ such that $a_{i_1} + \dots + a_{i_k} = 0$. Then, one can assume that $i_1 < i_2 < \dots < i_k$. Hence, $\pi = [s, a_{i_1}, \dots, a_{i_k}, t]$ is a path from s to t of weight $K = (n+1)P$.

For the converse, observe that every path from s to t (excepted the direct arc (s, t) of weight $(n+1)P + 1$) contains at least one arc that contributes to an element of A . Hence, a path of weight $(n+1)P$ describes a non empty subset of A with null sum. ■

Now, we consider DAGs without weights. We just count the length of the paths, i.e, the number of arcs in them.

Problem : Path of Length K .

input : a DAG $G = (V, E)$, two vertices s, t and an integer $K > 0$.

question : is there a directed path from s to t of length K .

This problem is solvable in deterministic polynomial time since one can assume that $K < |V|$ since G is a DAG. Hence, one can efficiently compute the matrix M^K where M is the adjacency matrix of G .

However, when one adds a supplementary condition, the problem becomes NP-Complete.

Problem 3 : Irreducible Path of Length K .

input : a DAG $G = (V, E)$, two vertices s, t and an integer $K > 0$.

question : is there an irreducible directed path from s to t of length K .

Here *irreducible path* means a sequence of vertices $[s = x_0, x_1, \dots, x_K = t]$ such that $(x_i, x_j) \in E$ if and only if $j = i + 1$. Of course, in linear orders, this problem is trivial since :

for $K \geq 2$, the answer is necessarily NO (by transitivity).

for $K = 1$, that corresponds to $s < t$.

For more general DAGs, this problem is non trivial anymore.

Proposition (3.) *The Irreducible Path of Length K Decision Problem is NP-Complete.*

Proof. Obviously again, it is in NP. For the NP Hardness, we reduce now the CNF SAT Problem to it. Let Φ be a CNF formula $C_1 \wedge C_2 \wedge C_3 \dots \wedge C_k$ where each clause C_i is a disjunction of literals on different variables. Assume that the total number of literals in Φ is N . We construct a DAG $G = (V, E)$ with $N + 2$ vertices : one vertex per literal in each clause C_i and two supplementary vertices s, t . The arcs are the pairs :

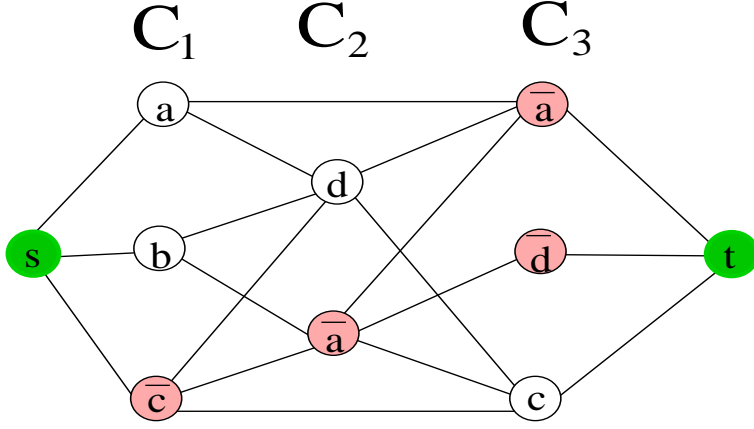
(s, x) for $x \in C_1$

(y, t) for $y \in C_k$

(x, y) for $x \in C_i$ and $y \in C_j$ with $j = i + 1$ and $x \neq \bar{y}$

(x, y) for $x \in C_i$ and $y \in C_j$ with $j > i + 1$ and $x = \bar{y}$

For $\Phi = (a \vee b \vee \bar{c}) \wedge (d \vee \bar{a}) \wedge (\bar{a} \vee \bar{d} \vee c)$, one obtains the following DAG :



The question is to find an irreducible path of length $K = (k + 1)$. This is equivalent to the satisfiability of Φ : one has to find at least one literal assigned to "true" in each clause (hence a path of length $k + 1$ from s to t) and with no contradictory assignments like $x = \text{true}$ and $\bar{x} = \text{true}$. For consecutive clauses, such an arc (x, \bar{x}) does not exist. Otherwise, this arc (x, \bar{x}) would make the path not irreducible.

■

- [1] S. BASAGNI, D. BRUSCHI, F. RAVASIO, *On the difficulty of finding walks of length k* , Theoretical Informatics and Applications **31** (5) (1997) 429–435.

Serge Burckel.
 INRIA-LORIA,
 Campus Scientifique
 BP 239
 54506 Vandoeuvre-lès-Nancy Cedex
 sergeburckel@orange.fr

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